1(a) If f is C^1 on (a,b), then f is Lipschitz on $[c,d] \subset (a,b)$.

Pf: Since f is C^1, f' is continuous on $[c,d]$.

Then there exists some $M>0$ such that $|f'(x)| < M$ for any $x \in [c,d]$.

For any $x, y \in [c,d]$, by Mean Value Theorem, there exists ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x-y).$$

Then $|f(x)-f(y)| = |f'(\xi)|\,|x-y| < M|x-y|$.

$\square$

(b) Let $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

Show that f is continuous at 0 but not differentiable at 0.

Pf: Since $|\sin \frac{1}{x}| \leq 1$, $-x \leq f(x) \leq x$.

Since $\lim_{x \to 0}(-x) = \lim_{x \to 0} x = 0$, by Squeeze Theorem,

$\lim_{x \to 0} f(x) = 0 = f(0)$.

Hence f is continuous at 0.

Note that $\frac{f(x) - f(0)}{x - 0} = \frac{x \sin \frac{1}{x}}{x} = \sin \frac{1}{x}$.

Take $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$.

Then $x_n \to 0$, $y_n \to 0$

and $\sin \frac{1}{x_n} \to 0$, $\sin \frac{1}{y_n} \to 1$ as $n \to \infty$.

Therefore $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Hence f is not differentiable at 0.

$\square$

2(a) Let $f \in R[a,b]$. Then there exists a sequence (φ_n) of step functions such that $\lim_{n\to\infty} \int_a^b \varphi_n(x)\,dx = \int_a^b f(x)\,dx$.

Rmk: Actually, you are expected to find such sequence (φ_n) of step functions with $\varphi_n \le f$. But due to the unclear statement of the question, $\varphi_n(x) = \frac{\int_a^b f(x)\,dx}{b-a}$ is also acceptable.

Pf: Since $f \in R[a,b]$, $\int_a^b f(x)\,dx = L(f)$.

By definition, there exists a sequence (P_n) of partitions such that $L(f, P_n) > \int_a^b f(x)\,dx - \frac{1}{n}$.

Write $P_n = (x_0^{(n)}, \dots, x_{m_n}^{(n)})$

and $\varphi_n(x) = \sum_{k=1}^{m_n} \inf\{f(x) : x \in [x_{k-1}, x_k]\} \chi_{[x_{k-1}, x_k)}$.

Then
$$\int_a^b \varphi_n(x)\,dx = \sum_{k=1}^{m_n} \inf\{f(x) : [x_{k-1}, x_k]\}(x_k - x_{k-1})$$
$$= L(f, P_n) > \int_a^b f(x)\,dx - \frac{1}{n}.$$

Clearly, $\varphi_n \le f$.

Then $\int_a^b \varphi_n(x)\,dx \le \int_a^b f(x)\,dx$.

By Squeeze Theorem, $\lim_{n\to\infty} \int_a^b \varphi_n(x)\,dx = \int_a^b f(x)\,dx$.

$\square$

(b) There exists a sequence (g_n) of continuous functions such that $\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b g_n(x)\,dx$.

Rmk: We shall prove $\exists\ (g_n)$ s.t. $\int_a^b |g_n(x)-f(x)|\,dx \to 0$

But $g_n(x) = \frac{\int_a^b f(x)dx}{b-a}$ is acceptable in the test.

Pf: We adjust the step functions φ_n in (a) to get continuous functions.

But (a), $\int_a^b |\varphi_n(x)-f(x)|\,dx$

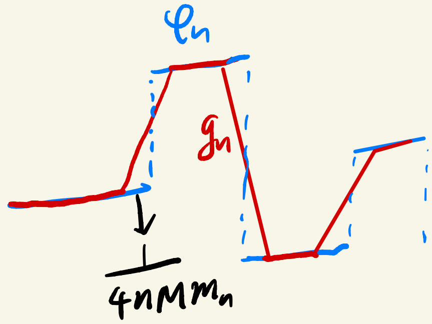
$$= \int_a^b f(x)\,dx - \int_a^b \varphi_n(x)\,dx \to 0 \quad \text{as} \quad n\to\infty$$

By Triangle Inequality,

$$\int_a^b |g_n(x)-f(x)|\,dx \le \int_a^b |\varphi_n(x)-f(x)|\,dx + \int_a^b |\varphi_n(x)-g_n(x)|\,dx.$$

It suffices to find continuous g_n such that

$$\int_a^b |\varphi_n(x)-g_n(x)|\,dx \to 0 \quad \text{as} \quad n\to\infty$$



$|\varphi_n| < M$

Take g_n as the piecewise linear functions like the red graph.

Then $\displaystyle\int |\varphi_n(x) - g_n(x)|\, dx \leq 2M \cdot \frac{2}{4nMm_n} \cdot m_n = \frac{1}{n} \to 0$ as $n \to \infty$.

Hence $\displaystyle\int |g_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.

3 (a) Let f_n, f be differentiable functions on (a,b)

Suppose $f_n(c) \to f(c)$ for some $c \in (a,b)$ and

$f_n' \rightrightarrows f'$ on (a,b).

Then $f_n \rightrightarrows f$ on (a,b).

Pf: Fix $\varepsilon > 0$.

Note that

$$|f_n(x) - f(x)| \leq |f_n(c) - f(c)| + |(f_n - f)(x) - (f_n - f)(c)|$$

$$= |f_n(c) - f(c)| + |(f_n' - f')(\xi)|\,|x - c| \quad \text{for some } \xi \text{ between } x \text{ and } c$$

$$\leq |f_n(c) - f(c)| + |f_n'(\xi) - f'(\xi)|(b-a).$$

Since $f_n(c) \to f(c)$ as $n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, $|f_n(c) - f(c)| < \frac{\varepsilon}{2}$.

Since $f_n' \rightrightarrows f_n$, there exists $N_2 \in \mathbb{N}$ such that for any $n \geq N_2$, for any $y \in (a,b)$,

$$|f_n'(y) - f'(y)| < \frac{\varepsilon}{2(b-a)}.$$

Take $N = \max\{N_1, N_2\}$

For any $n \geq N$, for any $x \in (a,b)$,

$$|f_n(x) - f(x)| \le |f_n(c) - f(c)| + |f_n'(\xi) - f'(\xi)|(b-a)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon .$$

Hence $f_n \rightrightarrows f$. on (a,b).

The converse is false.

$x^n(1-x) \rightrightarrows 0$ on $(0,1)$ (See Tutorial 10.)

$(x^n(1-x))' = nx^{n-1}(1-x) - x^n$

If $(x^n(1-x))' \rightrightarrows 0$, since $nx^{n-1}(1-x) \rightrightarrows 0$, we get

$x^n \rightrightarrows 0$ on $(0,1)$. Contradiction!

$\square$